THE HINDU ARABIC NUMERALS.

By A. A. Krishnaswami Ayyangar, Esq., M.A., L.T.

CHAPTER I.

The Ancient Numeral Systems of the World : General Principles.

A study of the early history of any branch of knowledge often throws light on many points which are apparently inexplicable and mysterious in the fully developed state of the subject presented to us in modern times. The modern decimal notation with its place-value scheme and its symbol for zero has passed through several vicissitudes at different times and in different lands before it attained its present simplicity, beauty and inevitableness, which conceal remarkably the intellectual throes endured and ineffectual circuitous paths gone through in the course of its inception.

Omitting the pictorial stage in which the picture of a thing was repeated as often as was required to represent its number, we shall take up the thread of development of the early representations of number at the symbolic stage, when strokes, vertical or horizontal, or a combination of both, began to be used. From very early times the fingers have served as a common aid to reckoning in groups of five and ten in almost all the countries of the world and have even suggested symbols for the representation of the fundamental numbers (1, 5, 10). Thus we have the stroke 1, suggested by the raised finger, used to denote unity by almost all the nations (Egyptian, Attic, Roman, Hindu and Chinese) of antiquity and the symbols V and X in the Roman notation suggested respectively by the hand with four fingers close together and thumb extended and the two hands interlinked together.

For representing intermediate numbers, *i.e.*, the numbers between unity and the group-numbers 5 and 10 two principles were devised, *viz.*, that of repetition and adjunction of symbols. But repetition soon reached its limit on account of the ocular incapacity to recognize immediately without counting the number of repetitions beyond (say) four; in some of the ancient notations such as the Babylonian where such repetitions were allowed upto nine, a suggestive form of arrangement was devised.

For example, in the Babylonian * symbol 1 for 89, the symbol for ten is repeated eight times and in two columns of four symbols each and the symbol for unity nine times and represented in three columns of three symbols each ; in the Attic and Roman notations, however, as well as in the Chinese and

* For the symbol, *vide* Appendix. For convenience of printing, all the symbols used in this paper are collected in the Appendix and numbered.

the Hindu, no symbol was repeated more than four times (occasionally five).

The practice followed in adjunction of symbols has been uniformly among all the early civilized nations of the world to write the number of higher denomination before one of lower denomination, according to the direction of the script in use.*

In fact, in the Hebrew notation [2] found on coins, the symbol for any number of higher denomination is written before one of lower denomination, since naturally in a right-to-left script, the symbol on the right is written before that on the left. We are also told that the folios in 'Tabula Registri de Visitatione Maneriorum per Robertum Decanum annodomini MCCXXII, given by Hale (Domesday of St. Paul's)† are numbered with Arabic numerals written originally from right to left, the numbers being afterwards struck out and a fresh series written in nearly the same character but from left to right; again in one of the manuscripts of the thirteenth century, ‡ the first thirteen quires are numbered "I", "II" .... "XIII" on the last page of each quire; then come 410 (=14), 510 (=15), 610 (=16) and so on.

Among nations like the Greeks and the Hindus writing the left-to-right script, the number of higher denomination is always placed to the left of that of lower denomination being written prior to it.

Examples of these are found in the Roman Notation[3], in the Attic[4] and Tamil notations[5], and in Nanaghat[6] inscriptions; whereas the notation in Kharoshti[7] numerals, (in use in N.-W. India in early times) is in keeping with the Kharoshti script which is written from right to left.

In the Egyptian notation, however, the numbers could be written either way, *i.e.*, from right to left or left to right, and in the former case the symbols were turned in the opposite way.

Whenever the above principle of adjunction is apparently violated in any numeral system, such adjunction has either a subtractive or a multiplicative significance. In Roman notation, as is well-known, a symbol preceding a higher one is to be subtracted, *e.g.*, $CM = -100 + 1000 = 900$. In the Hindu notation a symbol preceding a higher one multiplies it and thus has an adjectival force, as in the Tamil[8] notation; in the Kharoshti[9] system, the symbol of lower denomination coming to the right and thus preceding the other symbol in the right to left script, multiplies the symbol of higher denomination.

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* Many critics who speculate upon the origin of the modern notation forget this important fact that the terms 'before' and 'after' are always relative to the script in use.

† Quoted by G. F. Hall, on page 16 of his *Development of Arabic Numerals in Europe*, Oxford, 1915.

‡ P. 38. *Ibid.*

In the Babylonian[10] system, again, we have, similarly multiplicative adjunction. Sometimes the symbol with a multiplicative significance is written under or in close conjunction with another symbol; examples can be cited in the Attic[11] notation and in the Nanaghat[12] cave-numerals.

For representing intermediate large numbers the principles of the right and the left adjunctions as well as conjunctions are combined with that of repetition; examples are to be found in the Tamil[13] notation and in the Chaldaean[14] notation.

Even such a scheme as the above in which position plays a significant role, was not capable of representing large numbers in a compact and elegant form suitable for purposes of keeping accounts, etc., and so, another system of notation began to spring up soon.

According to T. L. Heath, the Greeks had the happy inspiration to conceive the original idea of using the letters of their alphabet for denoting the numbers of units, tens and hundreds that could occur in any number from 1 to 999. But before the Greeks, the Hebrews had possessed a system of numeration (about 500 or 600 B.C.) in Asia Minor, practically identical with the Greek alphabetic numeral system, and as we shall see later on, it was in the hands of the Hindus that this kind of notation was not only utilized to its fullest extent but a literary turn also was given to it. This Greek or Hebrew notation is to some extent similar in principle to the Brahmi notation in India as may be seen from the parallel examples in the Greek[15] notation and in the Brahmi[16] notation.

For expressing higher numbers the same alphabetic symbols were used (on a principle of periodicity) with such distinguishing marks[17] as dots, dashes, or bars placed over them to denote the number of thousands, etc., thus anticipating to some extent the use in the modern notation of the same symbol to denote different values according to position. Though position is not essential in this notation yet it follows the principle of placing the higher number before the lower as in the earlier iterative and additive notation. The thing that spoiled it was the use of separate symbols for tens and hundreds, which increased the strain on the memory though it led to as compact a representation as in the modern system.

There is a third system of notation known as the Babylonian sexagesimal system which also came so near to the modern one in the notion of positional value but diverged from it in the adoption of such a large base as 60 for numeration and in its failure to recognize the importance of the use of a symbol for zero. In an article by Cajori in the *American Mathematical Monthly* (January 1922) there is a reference to a Cuneiform Tablet (supposed to be as old as 2000 B.C.) which reveals the Babylonian operations with sexagesimal fractions similar to modern operations with decimal fractions. But the Babylonians had no mark to separate the fractional from the integral part, which was a serious defect. Thus the number 44 (26) (40) could be interpreted in an infinite number of ways and the correct interpretation could be judged only from the context.

It is believed that in this notation a sign was occasionally (not consistently) used to indicate a gap or the absence of any group or class; but it was not a part of the numeral system nor was it used in calculation. The Omicron '0' of Ptolemy was also not used as a regular zero but merely to represent blanks in sexagesimal fractions. It is a speculation of some historians of mathematics that probably with the introduction of the Babylonian sexagesimal fractions into India, passed also the principle of local value and the restricted use of the zero.

There was a kind of positional notation in vogue also among the Chinese [18] who used one set of numerals in the odd places and another set in the even places. In the Sun-Tsu Suan-Ching (of the first century A.D.) the arithmetical classic of Sun-Tsu we read, "In making calculations we must first know positions of numbers. Unity is vertical and ten horizontal; the hundred stands while the thousand lies; and the thousand and the ten look equally and so also the ten thousand and the hundred."

From the above brief survey it is evident that the different early systems of notations obtaining in different parts of the world contained the germs of the principle of the modern notation which was destined to develop in India where all these various strata in the growth of the notation are to be seen in a peculiarly indigenous form naturally leading to the place-value and the zero. What made the Greeks and the other nations who came so near the modern principle miss it is, in the present writer's opinion, their heterogeneous numeration which reckoned first in powers of ten upto one thousand and then in powers of one thousand, instead of regularly reckoning like the Hindus in successive powers of ten (एक, दश, शत, सहस्र, दशसहस्र, लक्ष, ...).

CHAPTER II.

The Development of the Numeral Systems in India: The Kharoshti and the Brahmi Numerals.

There were four different kinds of numerals in use in India from early times, *viz.*, the Kharoshti, the Brahmi, the symbolic word notation, and the alphabetic notation, before the decimal notation sprang up with the nine symbols and the zero. In this chapter, we shall describe the first two kinds which alone have some relation to the problem of the supposed Arabic origin of the modern numerals.

The Kharoshti * script which was in use in the North-West of India, was written from right to left and the Kharoshti numerals following the direction of the script were written, according to the usual practice, with bigger elements before (*i.e.*, to the right of) the smaller ones. These numerals occur in the so-called Saka inscriptions as early as the first century B.C. The fundamental signs are—

(i) one, two, three vertical strokes for 1, 2, 3 respectively.

(ii) an inclined cross[19] for 4.

(iii) a symbol[20] for ' ten '.

(iv) a cursive combination[21] of two tens for twenty.

(v) a sign resembling the Brahmi symbol with a vertical[22] stroke to its right for ' one-hundred '.

In this notation, unlike the Egyptian, not more than three repetitions are allowed of any symbol and a new symbol always springs up to avoid the fourth repetition. Thus ' 8 ' is represented by two four's[23]. A separate symbol[20] introduced for ' 10 ' and another[21] for ' 20 ' facilitate the writing of the numbers from 10 to 99, while the symbol[22] for ' 100 ' containing a multiplicative symbol on the right is necessary for representing numbers of three digits. The common principles underlying the structure of this notation and the Aramaic notations are so general that they could have suggested themselves to any one nation independently of another, while at crucial points, differences as well as similarities are noticed which make the theory of the ultimate Phœnician origin dubious. Julius Euting's Tables of the ancient Aramaic numerals have the Kharoshti symbols for 4 and 20 but the symbols for 10 and 100 are different. As regards the symbol for 4, even Bühler thinks it probable that both the Hindus and the Semites independently invented the cursive combination of the original four strokes.

Thus the Kharoshti numerals with their additive and iterative principles appear to be the first stage in the growth of the Hindu notation, corresponding to that of the ancient Egyptians and Babylonians. They are soon absorbed in and superseded by the more refined Brahmi notation in which one may hope to find the ancestor of the modern numerals.

The Brahmi notation is the most important of the early Hindu notations. There are several theories of its foreign origin, but none convincing enough. Some fragments of these numerals[24] occur in Asoka's Edicts as early as 300 B.C., and these are probably the earliest forms of our modern symbols. They reappear in the Nanaghat cave inscriptions of the second century B.C.

* The term ' Kharoshti ' means literally ' one having the ass's lip ' and therefore the notation may be either the invention of a sage with the ass's lip or the notation current among barbarians contemptuously termed by the Aryans as those having the ass's lip.

The inscriptions in the rock-cave up the Nanaghat hill contain 'a list of gifts made on the occasion of the performance of several yajnas and in naming the gifts, a kind of numerals is used differing in character from those hitherto found in West Indian Caves'.* To the perseverance of Pandit Bhagavanlal Indraji, whom Prof. E. J. Rapson refers to as the great Indian scholar whose memory is preserved in the British Museum by the shield which records his munificent bequest, we owe the copying and the elucidation of these çave inscriptions, which, though more or less obliterated, contain numerals in no less than thirty places. Some symbols[25] are the same as in the usual cave inscriptions; while others[26] resemble the figures found in the Nasik caves; the symbol[27] for '80' is the same as the one found in the coins of Virdama and Vijaya Simha Kshatrapa of Sourashtra: the symbols[28] for 100 and 1,000 are new: higher numbers are formed on the principle of multiplicative adjunction noticed to some extent in the Kharoshti notation: the smaller element occurring to the right of a bigger one forms a ligature[29] with it to denote the product of the two elements.

Regarding these Nanaghat symbols, Mr. Kaye says that they cannot be said to be well-established; for Bhagavanlal's interpretations of them are based on his *Akshara* theory which is not now generally accepted, and the abnormal symbols for 100 and 1,000 are not confirmed by any other sound examples.

The next evidence of the Brahmi numerals, we have in the Nasik cave inscriptions[30] in which the principle of the right adjunction of the smaller unit, as in Kharoshti numerals, with a multiplicative significance, is evident.

For further examples of the use of these numerals, the reader may consult E. J. Rapson's *Catalogue of the Coins of the Andhra Dynasty, W. Kshatrapa, etc.* (1908).

From the above description it may be evident that the Brahmi numerals belong to a non-place value system and have only a limited scope since they cannot represent large numbers. There is also some analogy with the Greek alphabetic notation in the fact that there are separate symbols for the different multiples of unity, and ten; unlike the Greek notation, the symbols for multiples of 100 and 1,000 are formed on the principle of multiplicative adjunction. The idea of representing any number less than 10^n by $9n$ or less symbols (n being any integer) is such a fundamental one that it could have suggested itself independently to any intelligent nation and it is obvious that the Hindus who managed with 20 primary symbols the representation of all numbers less than 10^5, could not have obtained any suggestion from the

* *Vide* Pandit Bhagavanlal Indraji's article entitled "On Ancient Nagari Numeration from an Inscription at Nanaghat", *Journal of the Bombay Branch of Royal Asiatic Society*, 1876, Vol. XIII, p. 404.

Greeks for this kind of notation. (*Vide* G. H. Ojha's *The Palæography of India*, pp. 103, 114.)

The first four symbols of the Brahmi notation are apparently derived from the corresponding symbols of the Kharoshti numerals by turning them through an angle, a frequent process in the transformation of numerals, easily accounted for by the psychological fact* that the primitive or the less developed minds cannot recognize the configuration and orientation of a symbol as an essential feature of the notation. According to Kern, the device of indicating the number 4 by a cross is so natural and ingenious at the same time that any comment on it may be superfluous, and all the latter forms of '4' are off-shoots of this ancient sign. But Mr. G. R. Kaye doubts this conclusion, since all the early examples except one are markedly differentiated from it. He does not believe in the derivation of 5 from 4 and, indeed, he says that no principle of formation of the symbols from 4 to 30 can, at present, be offered; but possibly the symbol for '40' is derived from that for '30' by the addition of a stroke, while the 'sixty' and 'seventy' as also 'eighty' and 'ninety' appear to be connected similarly. He also gives us a warning that the principle of formation in this case appears more marked in the later symbols and we must be careful about forming any definite conclusion as to the origin of the system from such evidence.

There are several theories regarding the foreign origin of the Brahmi numerals. For instance, Bayley assumes that the Hindus must have borrowed from four or five different, partly very ancient and partly modern, sources; and Burnell points out the general agreement of the principles of the Indian system with those of the Demotic notation of the Egyptians and comes to the provisional conclusion that the South Indian Asoka system is derived from the Egypto-Phœnician system of numerals; Bühler and Barth concur in Burnell's view. In this connection Pandit Bhagavanlal Indraji's *Akshura* origin of the numerals deserves mention, which, if accepted, would show that the numerals were a purely indigenous development. We are told that the books of the Nepalese and the North Jains have their pages numbered by a singular series of letters.

ए 1	ख 4	ग्र 7	मृ 10	
द्रि 2	तृ 5	हृ 8	थ 20	सु or सू 100
त्रि 3	फ्र 6	ओ 9	ल 30	

* In teaching children of about 5 years to write the letters of the alphabet, I have often met with such inversion of the letters, for example a child may write 'B' in any one or all of the following forms:— ᗺ, ᗰ, ᗯ.

The origin of this kind of notation is still obscure though there are curious survivals of its usage even in modern times (*vide J. R. A. S.*, 1896, C. Bendall's article: 'On a System of Letter-Numerals used in S. India'). The Pandit, observing some points of similarity between these *Akshara* numerals and the Brahmi symbols formulated his famous *Akshara* theory; but it has been rejected by Burnell, though partially accepted by Bühler. According to Bühler, the signs have certainly been developed by Brahmanical schoolmen, since they include two forms of उपध्मानीय which, without doubt, have been invented by the teachers of the Siksha.

Besides the above there are other conflicting theories which are as fanciful as they are absurd. While it is believed that several eastern nations have invented independent systems of numerals of their own, why should we seek, in vain, to find an extraneous origin for the Indian numerals alone and get landed in unsatisfactory hypotheses? The very fact that, of the scores of hypotheses that have been trotted out to trace them to a foreign source, none have come anywhere near the truth is sufficient proof to show that no such foreign source really exists. Whatever be the origin of these symbols, the symbols, as they were, had no special virtue in them, (that one should attempt to trace them to a foreign source and thereby to deny the credit of invention to the Hindus), except for the fact that the zero symbol came to be introduced into it later on and that the modern place-value system developed in it absorbing nine of its symbols and rejecting the rest.

The Brahmi symbols are the ancestors of the so-called Arabic numerals. It does not, indeed, require so much imagination to perceive the resemblance between them and the modern numerical symbols* as to derive them from the Greek or Arabic forms which have sometimes to be turned round or turned over or even distorted so that they may lead to the modern symbols. In a recent article (by an F. R. S.) in the *Mathematical Gazette*, July 1925, we read such statements as the following; many of these are quite untrue being probably based on such authorities as Mr. Kaye.

"Moreover recent research has thrown some doubt on the antiquity of Indian mathematics. The evidence on which we largely depend in this connection is a Hindu treatise on Astronomy called the *Surya Siddhanta*, which was probably composed in about A.D. 500 and which seems to have derived a great deal from the Alexandrian school. ----

There is no necessity to suppose that the Arabic numerals were derived from the Greeks through the Hindus. They may well have arisen in the Near East itself.

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* *Vide* the Brahmi symbols and the modern symbols given in parallel columns in the Appendix numbered (30).

After the collapse of the Roman Empire, the Arabs inherited the scientific traditions of Alexandria and there is no doubt that they must, through the writings of Ptolemy and others, have been made familiar with the Greek numeral symbols.

Now it is very remarkable that these symbols (at any rate as regards 6, 7, 8, and 9) bear a singular resemblance to the corresponding Greek letters.[31] Thus, the late forms of Vau (capital and cursive) are 'C, S' and the latter is almost identical with the sign for '6'.................4 is a serious difficulty. It is true that there is an old form[32] which leads at once almost to 4; but unfortunately, the intermediate forms do not bear this out. An early Indian symbol is one[33] which does look like a four stroke symbol: this becomes in later Indian script ४, whose variants are .the Arabic symbol[34] and the European twelfth century symbol.[35]

But the decisive argument is really supplied by the *zero*. This is absent from the early Indian scripts, or else zero is denoted by a dot. On the other hand, we have definite evidence that '0' is used to denote zero in Ptolemy's Almagest which the Arabs had thoroughly mastered.

If this argument is correct, the Arabic numerals have really been derived from the Greek alphabetic numerals * by omitting the separate signs for tens and hundreds and by importing three new signs for 1, 2, and 3.

In attempting to trace the source of the modern numerals, it is futile to associate them, on account of some fancied resemblance, with the notations developed in countries like Greece, Asia Minor or Arabia, where there was no such indigenous systematic development of positional value nor any systematic use of the zero as we find in India. The story of the parallel development of a positional notation in India, alongside of the non-positional one, we shall consider in a separate chapter.

CHAPTER III.

The Development of the Numeral Systems in India: The Symbolic Word Notation and the Alphabetic Notation.

Nowhere among the other ancient nations of the world do we find such a consistent scheme of numeration as among the Hindus, which naturally reflected itself in the later place-value system. The early Hindus counted regularly in the ten-scale as so many units, tens, hundreds, and so on in successive powers of ten, unlike the Greeks, the Arabs, the Chinese, and the Japanese who introduced the thousand in the middle of their scheme of

* To perceive how baseless this theory is, we have only to note, that out of ten symbols in the modern notation, only four (*i.e.*, 40%) have some fanciful resemblance to the corresponding Greek alphabetic numerals and three can, with a good deal of strain, be made to resemble the Greek numerals while the rest are admittedly importations from a non-Greek source.

numeration which was really a step away from the decimal scheme. While the English eleven and twelve are out of harmony with the later teens, the Sanskrit numeration has एकादश, द्वादश, etc. (one and ten, two and ten and so on). This early Sanskrit numeration in which large numbers had to be expressed in a periphrastic way such as अष्टोत्तरशत (eight above hundred), later took a more convenient form, whereby merely the numbers of the units, tens, hundreds, etc. occurring in a number were mentioned in regular order in the increasing scale of powers of ten, the names of the powers being omitted. Thus five, seven and two meant five units, seven tens and two hundreds. This scheme naturally necessitated the explicit mention of the absence of a particular power of ten in any number and the word शून्य came to be introduced to denote such gaps. To adapt this numeration to the verses in which, generally, the early Hindu scientific works were written, a kind of vocabulary was devised, which is well described by Brahmagupta thus :—

"If you want to write one, express it by everything which is unique as the earth, the moon ; two by everything which is double as for example black and white ; three by everything which is three-fold, the nought by heaven, the twelve by the names of the Sun."*

Mr. G. R. Kaye rejects Alberuni's statement that Brahmagupta invented this notation known as the word-numeral notation (भूतसङ्ख्या objects denoting numbers) and assumes, without any proper authority that it was probably introduced into India from the East. In this connection, it will only be proper to inquire whether in the East there has been any such extensive use of this notation as is found abundantly in the Indian astronomical and mathematical works beginning from about the middle of the sixth century A.D.

The earliest epigraphical instance of the usage of this notation in India proper dates 867 (गिरिरसवसु) Saka samvat, but in Cambodia Sanskrit inscriptions are found belonging to about 600 A.D. The period of invention of this system is uncertain and the earliest trace, as noted by Weber, seems to be in the *Srouta Sutra* of Katyayana or Latyayana. Numerous examples occur also in Pingala's manual of metrics. Varahamihira of the sixth century A D. uses this notation in *Brihat-Samhita* (*J. R. A. S.*, Vol. I, N. S., p. 407). Aryabhata might have known it and probably had tried to improve on it in

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* In this statement of Brahmagupta, we find a remarkable anticipation of the line of ideas which led to the modern definition of number (serving for finite collections) given in Bertrand Russell's *Introduction to Mathematical Philosophy* (Chapter I, pp. 18, 19).

'We may now go on to define numbers in general as any one of the bundles into which similarity collects classes.....................: In other words, a number (in general) is any collection which is the number of one of its members ; or more simply still : a number is anything which is the number of some class.'

his alphabetic notation, just as the later Aryabhata, who certainly knew this word-numeral notation, thought of substituting in its place his scheme of कटपयादि notation which is a complex of the alphabetic and the decimal notation and combines the advantages of both.

Dr. Bühler thinks that the Dwandwa compounds containing words with numeral significance presuppose the existence of the decimal notation, especially when these compounds had to be dissolved by 'and'. But I should think that the word numeration suggested the place-value scheme and the decimal notation, when the words had to be translated into symbols. Such symbols were supplied by the non-positional Brahmi notation, which was current side by side with the positional numeration. Thus it must have flashed to some genius (whose name may remain unknown for all time), that the positional numeration and the non-positional symbolic notation could be welded together into the simple and beautiful scheme of the decimal notation. I am disposed to believe that the positional numeration served in India the same purpose as the abacus in Rome, China and Japan to suggest the place-value principle and the zero. In fact, the positional numeration when translated into writing, naturally leads to a form of abacus and there are evidences of which we shall speak at some length in the next chapter, of the existence of such a form of abacus in popular use in India.

Before proceeding to discuss the decimal notation in India, we shall take up an interim short-lived development of a kind of ingenious alphabetic notation * due to Aryabhata. Dr. Fleet seems to think that the suggestion for this notation might have come from the Greek alphabetic notation, while Mr. G. R. Kaye calls it a crude adaptation of the Greek plan. But we believe that Aryabhata must have got his inspiration from the greatest grammarian of the world, Panini, who was the first Indian, probably, to conceive of denoting numbers by the letters of the alphabet in their order (*vide* † Goldstucker's 'Panini', p. 44). Aryabhata's notation illustrates one method of adapting the decimal numeration to symbolism. The positional principle was there, but utilizing the position itself for indicating value was not yet thought of and hence a temporary arrangement was devised to indicate the positional value by vowels, the consonants denoting the numerals proper,

e.g., ङि शि बु ण्लृ रव्णृ = 1582237500.

The object of this system was conciseness which was certainly achieved and its formulæ are far more compact than in any other system of notation.

* For a detailed explanation of Aryabhata's scheme, *vide J. R. A. S.*, 1911, pp. 109-126.

† Goldstucker's reference to Patanjali and Katyayana about Panini's using letters in his Adhikara rules for the notation of numeral values, is rather dubious and the present writer is unable to locate the actual reference in Patanjali's *Mahabhashya*. *Cf. The Palæography of India* by G. H. Ojha, p. 124.

According to Dr. Fleet, this system implies the use of a board ruled and lettered in some such manner as in the figure below, but otherwise left blank for resolving the details of any particular statement :—

ऌ		ऋ or ॠ		उ or ऊ		इ or ई		अ or आ		
							5			ङि
						7				शि
				2	3					बु
1	5									ण्ऌ
		8	2							ख्यृ
1	5	8	2	2	3	7	5	0	0	ङि शि बु ण्ऌ ख्यृ

The question remains open, however, whether the ancient Hindus disposed of the blank spaces either by dots or by leaving them blank or otherwise.

Since this notation was too learned and difficult for the average man, it was soon forgotten and even Lalla, Aryabhata's earliest disciple, abandoned it in favour of the more popular numerical words which could more easily be remembered and fit in with metrical euphony. But it must be remembered that Aryabhata's alphabetic notation marks an important stage in the development and is a necessary precursor of the Indian decimal notation, in that it suggested a method of using the same symbol, say, क, with such variations as कि कु, etc. to denote a multiple of a power of hundred. What remained yet to be done was to drop even the vowels and make the position itself indicate what power of ten is intended. This would require setting apart a consonant for zero also. But it probably took many long centuries to recognize that the *zero* was also a numeral on a par with the other numerals and that a separate symbol was necessary to denote it. It was only in the eleventh century, after the decimal notation with its place-value and zero had become definitely established that the alphabetic notation was thought of once again and re-adapted to the new notation. In this connection it is worthy of remark that the alphabetic notation in India was felt more or less as a necessity owing to the exigencies of metrical composition and therefore there is a greater likelihood of its being indigenous to India than a casual loan from Greece or elsewhere.

(To be continued.) xix · 2-9-40.

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(Continued from Vol. XVIII, No. 4.)

CHAPTER IV.

The Development of the Numeral Systems in India: The Decimal Notation. The Abacus and the Symbol for Zero.

ONE noteworthy feature of the development of the numeral notation in India is its progressive continuity* and growth—one system leading on to the next and getting itself absorbed in it, imbibing new life partaking the essential principles of the old and the new. We have seen how the iterative and additive notation of the Kharoshti numerals lent as it were its first four symbols to the Brahmi notation and got merged in it. Again, the Brahmi numerals did not advance further than a few hundreds, since the word-numeration developed alongside of it with the place-value principle and arrested the growth of the non-positional notation. Otherwise, we should have had, even in India, a kind of extension of the non-positional notation with a periodic principle corresponding to that of the alphabetic notation of the Greeks with the dashes and dots for numbers greater than 1,000. Witness also the two-fold alphabetic notation, one before the invention of the decimal notation and another after it, utilizing the positional principle and the zero. There has been also similarly a two-fold word-numeral notation, one non-positional and the other positional distinguished by the way in which the Dwandwa compounds (containing the numeral names) were dissolved, the one by 'or' and the other by 'and', the latter presupposing the existence of the decimal notation (*vide* Buhler's *Indian Palæography*).

When the decimal notation with its nine figures and a symbol for zero was actually invented is a matter enveloped in deep mystery. When the word-numeral notation was in full swing, the Brahmi symbols were ready at hand to be utilized for the purpose and only one new symbol had to be invented, that is, for zero. The word शून्य or its metaphorical equivalent आकाश (the spherical vault of the heavens) denoting the absence of a power of ten in the word-numeration should have easily suggested the symbol '0'. Probably an earlier or an alternative form of this symbol is the dot symbol mentioned in Subhandu's *Vasavadatta* and also in the *Bakshali Manuscript*.

* There is nothing like it in the notations of other nations; for example, there is hardly any point in common between the earlier Attic notation and the later Greek alphabetic notation.

In Subhandu's *Vasavadatta*, we read—

विश्वं गणयतो विधातुः शशि कठिनी खण्डेन तमो मषीश्यामे अजिन इव वियति संसारस्यातिशून्यत्वात् शून्यबिन्दव इव विलिखिताः ।

Here the author brings together in a suggestive simile, mathematics, poetry and philosophy in a truly Indian fashion; the passage bears an important testimony to the zero-symbol in vogue in the author's time, which has been fixed by scholars to be probably between 540—570 A.D.

Again, in the *Bakshali Manuscript* whose date has been variously fixed between the fourth and the tenth centuries A.D., there is a clear mention of the decimal notation with nine symbols[36] * and the *dot* zero. (It is very curious that these symbols bear some analogy to the Kanarese and Telugu numerals.)

In Vyasa's Commentary upon Patanjali's *Yoga Sutras*, the characteristic feature of the position value of the numerals in the decimal notation is brought in by way of elucidating a philosophical point.† We quote below from Ramaprasada's translation of the Bhashya of Patanjali's *Yoga Sutra* (Chapter III, 13) :—

एतेन भूतेन्द्रियेषु धर्मलक्षणावस्था परिणामा व्याख्याताः ।

'It is not the characterized object that is possessed of the three paths of being. It is the characteristics that are possessed of the three paths. They may be visible or latent. Of these the visible ones assume different conditions and are termed accordingly differently, because the conditions are different and not the substance. This is in the same way as the figure of '1' means '10' in the tens' place, '100' in the hundreds' place and '1' in the units' place. Or again, a woman although one is called a mother, a daughter, and a sister......'

The *Vyasa Bhashya* cannot have been composed later than the sixth century A.D. The decimal system was therefore known to the Hindus long before it appeared in the writings of the Arabs or Græco-Syrians.

The Hindus called the decimal notation अङ्क पल्ली, the word अङ्क literally meaning a mark or a symbol. In the word-numeral notation adopted by Varahamihira (sixth century A.D.) and others, the word अङ्क is used to express the numeral 9 (*vide Panchasiddhanthika*, 18, 33) signifying that nine and not ten numerals were in common use. Probably the symbol for zero had not then been invented (or though invented, not recognized as a numeral); but by the ninth century at least, all the ten symbols should have been perfectly well established. Thus it becomes significant that in the prefatory

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* *Vide* Appendix to this article, in Vol. XVIII, No. 4, of this *Journal*.

† The discovery of this reference in *Vyasa Bhashya* is due to Dr. Sir Brajendranath Seal who has also discovered similar references in Buddhist authorities earlier than sixth century A.D., which Prof. Scherbatsky has now traced. These references alone are sufficient to settle finally the priority and the originality of the Indian notation.

chapter of Mahavira's *Ganita Sarasangraha*,* a work of the ninth century A.D., there occurs a list of equivalents, in the word-numeral system for only ten numbers *viz.*, the numbers from 1-9 in order and lastly zero. Mahavira did not think it necessary to give the equivalents for higher numbers as they were obviously superfluous in the decimal notation.

In connection with these ten numerals of the new notation, a rule sprang up (the date of which is unknown), 'अंकानां वामतोगतिः' that is, the order of the numerals is from right to left. We do not know whether this rule refers to the order in writing or to any arrangement adopted in some form of abacus in use in early times. Mr. Dikshit tells us (*vide Indian Antiquary*, XX, 54) that Hindu astrologers were using a wooden calculating board called पाटी and hence the name पाटी गणित for Arithmetic. Warren in his *Kalasankalita* makes mention of Indian almanac makers computing eclipses, scoring their quantities with shells instead of writing them in figures. Unfortunately none of these practical methods of computation have been recorded in any of the known Hindu arithmetical treatises. This is probably to be expected, since the treatises are intended to supply only the theoretical and scientific basis for the practical methods of computation while the mechanical methods of computation with shells, etc., were probably handed down orally from generation to generation.

In Gow's *History of Greek Mathematics*, we find the dictum 'The Cipher is yet to be invented before the abacus can be discarded'. Since there is reason to believe that the Hindus generally reckoned on a board covered with sand, and the symbol for zero was invented probably some time later than the other nine symbols which were directly taken from an earlier non-positional notation, some palpable aid to reckoning like the abacus may have been in vogue (in accordance with Gow's dictum) in the transitional period, *i.e.*, from the quasi-positional to the definitely positional notations. Further, an actual need for such an adventitious device as some form of abacus may have been felt by the early Hindu astronomers, who at least from the fifth century A.D. began to calculate with huge astronomical constants.

The essence of the abacus was the arrangement in columns which were marked off by lines and allocated to the successive denominations of the numerical system in use. The number of units of each denomination was shown in each column by means of pebbles, buttons, or the like. We have hardly any details of the Indian abacus as we have of the Chinese swan-pan, the Japanese soroban, and the Roman abacus, though the use of a tray strewn with sand and the use of pebbles to reckon with have been attested by many

* *Vide* pp. 6, 7 of the Sanskrit Text of the *Ganita Sarasangraha* of Mahaviracharya edited by M. Rangacharya, Madras, 1912.

writers. There is a curious parallel to this state of things among another intellectual nation, the Greeks; for we learn from Dr. T. L. Heath that there is very little evidence as to the actual use of the abacus in Greece. Probably the abacus with its 'tableau colonne' is an invention of a less mathematically gifted race, and a sort of mental abacus must have sufficed for the Greeks or the Hindus. Indeed even in modern times, some such device as the abacus is employed in schools to explain the place-value notation to young children.

Some light is thrown on this question of the use of the abacus in India by Dr. Fleet in his article "The Use of the Abacus in India" (*J. R. A. S.*, 1911). He draws attention to the following passage which perhaps belongs to the first century A.D :—सभूरिको गणित्रे (कीर्तितः ?) श्वेतवर्णं गृहीत्वा गणयितुमारब्धः ।

May not the word गणित्र in this passage, on the analogy of खनित्र, लवित्र, etc., mean an instrument such as some form of abacus to reckon with?

One of Mr. Kaye's *a priori* postulates is that the value of position and the invention of zero were so obviously derived from the use of the abacus. At any rate, in India, the abacus need not have led to the zero but rather the peculiar Hindu system of numeration which gave, in order, the number of units, tens, hundreds, etc., in a number.

From very early times, the Sunya had acquired a special significance in India, not found in the Greek or other ancient arithmetics of Europe. Brahmagupta, living in early seventh century, treats of the results of the four fundamental operations with zero and the *Ganita Sarasangraha* of the early ninth century gives similar discussions of calculations with zero. Probably the zero must have been perceived even in the early stages of Arithmetic as a result of subtracting a number from the same number; and very likely the ideas of धन, ऋण and शून्य (+, — and 0) flashed to the Indian mind simultaneously, being suggested by the familiar fact that a man becomes wealthy by spending less than what he earns, or indebted by spending more, or *penniless* (as suggested by the words रिक्तहस्त, or शून्यहस्त) by living from hand to mouth (*i.e.*, spending all he earns). Besides, the Sunya displayed an important rôle in Indian Philosophy which preached incessantly the Maya or the emptiness of the world. Smith and Karpinski have well said, regarding the Indian invention of the zero, that this making of nothingness the crux of a tremendous achievement was a step in complete harmony with the genius of the Hindu.

There is also another circumstance which emphasizes the fact that the zero or the dot was originally used by the Hindus for any kind of blank. In the *Bakshali Manuscript*, the dot symbol for zero is used to denote the unknown or absent quantity* as well as zero. This shows the Hindus'

*An analogous use of the zero, for the unknown quantity in a proportion, appears in a Latin manuscript of some lectures by Gottfried Wolack in the University of Erfont in 1467 and true insight into the purpose of the symbol, *viz.*, to denote any absent or non-existent quantity, whatever the absence or the non-existence may be due to. That the Arabs did not understand the uses of zero is borne out by the occasional use by Al Battani, a famous Arabic astronomer, of the Arabic negative *lā* to indicate the absence of minutes or seconds. There is also evidence that many writers in Europe were using the symbol for zero in some form or other between the twelfth and the fourteenth centuries, but without understanding its true import.

What suggested the form for zero is purely a matter of speculation. The dot symbol was frequently used by the Hindus to fill up gaps in their manuscripts and so might have been thought of for the purpose of indicating also an absent quantity in mathematics. Smith and Karpinski inquire whether the fact that the early European Arithmetic following the Arab custom always put the '0' after the nine symbols 1-9, suggests that the smaller circle 'o' was derived from the old Hindu symbol,[37] a spurred circle for ten. The popular Indian use of the formula $\sqrt{10r^2}$ (where r is the radius) for the area of a circle may also be significant in this connection. Again, from Dr. Shama Sastry's thesis on the origin of the Devanagari alphabet (*Indian Antiquary*, 1907, p. 22), we learn that the rôle of the dot symbol (बिन्दुर्वर्तुलाकारः) is very prominent in the Tantric Hieroglyphics, which in the Doctor's opinion may have been the basis of the Sanskrit alphabet. May I suggest that the Tantric dot symbol was probably pitched upon by the Hindus for their last numerical symbol even as the visarga, the double dot (:), marked the close of the vowel series ? The circle symbol in India replaced the earlier dot symbol, which came to be used later in other contexts. In *Bhaskara*, we find the dot above a number indicating that it is negative.

The name of this all-important symbol deserves a little notice. It is commonly accepted that the Hindu 'Sunya' passed over into the Arabic as as-sifr or sifr, which Leonardo called zephirum in his book *Liber Abaci* on the Hindu numerals. Maximus Planudes, writing under the influence of both the Greeks and the Arabs called it 'tziphra'. In the Italian arithmetic of the fourteenth century, it became zeuro and çeuro, which led to the modern zero. The English cipher and the French chiffre are also derived from the same Arabic word as-sifr. Owing to the resemblance of the circular form '0' to the Greek letters Theta and Omicron and also to several objects such as the wheel and the circular iron used to brand thieves with in mediæval times, the symbol was also called by such names as 'theca', 'Omicron', the wheel, circulus, etc. But the common name was, of course,

1468. The usage was noted even as late as the eighteenth century.—*The Hindu Arabic Numerals*, by Smith and Karpinski, pp. 53, 54.

cipher. Wallis, in his *Opera Mathematica*, was one of the earliest to discuss the derivation of this word, giving the following variations ziphra, zifera, siphra, ciphra, tsiphra, tziphra, etc.

The Katapayadi Notation.

Not content with the decimal notation with its nine numerals and zero, the Hindus, with their usual fondness for the alphabetic notation on account of its singular adaptability to literary form, soon began to revise their old notation in the light of the new positional invention.

In the *Mahasiddhanta* of Aryabhata (II) a work of the twelfth (?) century, the notation is defined thus :—

रूपात् कटपयपूर्वा वर्णा वर्णक्रमाद्भवन्त्यङ्काः ।
न्ञौ शून्यं प्रथमार्थे आ छेदे ऐ तृतीयार्थे ॥

According to this notation, the consonants were given values as follows :—

1	क,	ट,	प,	य
2	ख,	ठ,	फ,	र
3	ग,	ड,	ब,	ल
4	घ,	ढ,	भ,	व
5	ङ,	ण,	म,	श
6	च,	त,		ष
7	छ,	थ,		स
8	ज,	द,		ह
9	झ,	ध,		ळ
0	ञ,	न.		

In this notation, the numeral letters read from left to right and not from right to left as in the word-numeration. This shows that the scheme is based on the decimal notation and not upon the old numeration.

Example :—The number of revolutions of the Sun (*i.e.*, of the earth, really speaking) in a Kalpa is

घ	ड	फे	न	ने	न	न	नु	नी	नाः
4	3	2	0	0	0	0	0	0	0

Here the vowels have no significance. This latitude as well as the alternative consonants for the same number are intended to satisfy the exigencies of metre and metrical euphony.

There is another modified form of the above notation which secures better metrical euphony and literary effect (than Aryabhata's system) by its pun and other suggestive associations. It serves also as a better mnemonic, not being a jumble of artificial syllables but a set of significant words suited to the context. The dates (in Saka Samvat) of birth and death of the famous religious leader of South India, Sri Ramanuja, are given by the suggestive phrases :

धीर्लब्धा = 939 (Wisdom is born (achieved))

धर्मोनष्टः =1059 (Virtue is dead)

In this scheme, the last letter alone of a conjunct consonant has a numerical significance and the letter numerals read from right to left as in the word-numeration. This embodies all the fundamental principles of the earlier notations and illustrates, once again, the characteristic continuity in the Indian development.

Mr. Whish in the *Transactions of the Literary Society of Madras*, Part I, 1827, p. 60, has cited a work entitled Sadratnamala, as telling us that the circumference of a circle whose diameter is one parardha or 10^{17} is expressed in the above notation by

भद्राम्बुधि सिद्धजन्मगणित श्राद्धास्म यद्भूतपर्गाः ।

(3.14159265358979324).

This shows us the possibilities of this kind of notation. The notation in the above form is very popular, even now, in India and Burma.

One of the advantages of the alphabetic notation which was early recognized by the Hindus and which accounts, to some extent, for its popularity is that it does not admit easy alteration, as the figures do, since any change would affect the sense as well as disturb the metre.

CHAPTER V.

The Claims of the Arabs to the Invention of the Modern Numerals : The March of the Numerals to Europe.

In his *Indian Mathematics*, a very misleading work on the subject, Mr. G. R. Kaye hints at the Arabic origin of the numerals in the following words :—

'Further, there is evidence that indicates that the notation was introduced into India, as it was into Europe, from a right to left script.'

We elsewhere refute Mr. Kaye's arguments regarding the derivation from a right to left script and content ourselves here with the remark that such arguments as his would apply quite as well to the Roman and Greek systems as to any other.

Before Mr. Kaye, such influential writers as Tartaglia in Italy and Koebel in Germany had asserted the Arabic origin ; but the Arabs themselves never laid claim to the invention and there was, indeed, for a long time, a struggle among them between the Hindu numerals and the indigenous Arabic ones, just as there was no love lost between the algorists and the abacists in Europe in the middle ages. (This is a clear evidence to show that the Hindu numerals were foreign to these lands.) We learn from Ali ibn Ahmed al-Nasawi's arithmetic of *c.* 1025, that the number-symbolism was still undecided in his day, most people preferring the strictly Arabic forms. Besides, the Arabs had no number names beyond one thousand and it is very unlikely that such a nation could invent a place-value system.

It is not known, however, when the Arabs really came across the Indian numerals. We are told that about 156 A.H. (772 A.D.) during the reign of one of the Abbasides, an Indian traveller brought to Bagdad a treatise on Arithmetic and another on Astronomy and that these treatises have been translated into Arabic. Probably the Indian numerals were introduced among the Saracens at this time along with the Astronomical Tables. Before this time, the Arabs had no numerals. They were writing numbers in words, and in some places, adopted for convenience the notation of the conquered lands. They had also an alphabetic notation on the analogy of the Greek system. But when the Hindu notation once stepped in, its advantage over the other systems was immediately recognized and it soon became popular with merchants and arithmetical writers. For over five hundred years Arabic writers and others continued to call their works on Arithmetic, ' Indian ' or ' Hindu'.

The first Arabian writer to whom the world owes its first algebra, Mohammed ibn Musa Alkhowarizmi of the eighth century distinctly acknowledges in his Arithmetic the debt to the Hindus in the matter of the numeral notation. The Arithmetic of Khowarizmi 'excels all others in brevity and easiness and exhibits the Hindu intellect and sagacity in the grandest inventions.' So says an Arabic writer (*vide* p. 102, Cajori's *History of Mathematics*, 1919).

In early eleventh century, Alberuni who is considered as a phenomenon in the *History of Eastern Learning and Literature*, refers to the Hindu numerals अङ्क, occurring in different shapes in different parts of India. In his *Chronology of Ancient Nations* translated by Sachau, we find (on p. 64) the following statement :

' If we reduce this cycle of 19 years (*i.e.*, 6939 days 16 $\frac{595}{1080}$ hours) to fractions and change it into halaks, we get the following sum of halaks :—

179, 876, 755 expressed in Indian ciphers.'

A man of Alberuni's reputation, who had much of the modern spirit and method of critical research, would not have blindly believed in mere tradition and attributed the new numerals with the place-value system to Hindu sources.

Again in the first half of the fourteenth century, we find that Maimus Planudes, a French monk, following the Arabic custom, called his work Indian Arithmetic. So late as the sixteenth century Baha Eddin, the writer of a compendium of Arithmetic *Kholaset al-hisab*, says: 'Learned Hindus have invented the well-known nine figures for them.' Another interesting, though bizarre, reference to the Brahmin origin is that of the Arabic astrologer Aben Ragel of the tenth or eleventh century. He held that the Brahmins derived their numerals from the figure of a circle with two diameters.

From numerous such evidences as the above in the *History of Arabic Literature and Mediæval European Works* based on Arabic learning, Smith and Karpinski are able to conclude forcibly that the Arabs from the early ninth century onwards fully recognized the Hindu origin of the numerals. We shall discuss in the next chapter the opposite view held by Mr. G. R. Kaye who asserts that the Arabic words 'hindi', 'hindisa' and 'hindasi' have been misinterpreted as 'Indian' and that the mediæval references to India do not indicate 'India proper' but often simply 'the East'.

One important evidence tending apparently to support the non-Hindu origin of the modern numerals is the total absence of any reference to them in the arithmetical works of some of the eminent Arab mathematicians of the tenth century. Abu'l-Wefa (940-998 A.D.) wrote an arithmetic which entirely ignores Hindu numerals. Alkharki, an Arabian algebraist of the 10-11th century wrote an algebra under the title *Al Fakhri* which contains an exposition of the methods of Dipphantus and little whatever of Hindu Indeterminate Analysis; an arithmetic by the same author, again, is constructed wholly after the Greek pattern and excludes the Hindu numerals. According to Cantor and Heath, there were probably two schools, one of which favoured the Greek and the other the Indian methods.*

According to Smith and Karpinski, the Arab, by himself, never showed any intellectual strength and they give this as one of the reasons for not ascribing too much credit to the purely Arab influence. But, when the Arab culture joined the Persian and an empire was set up at Bagdad, which enjoyed a favourable position, more or less midway between the two great old centres of scientific thought,—Greece in the west and India in the east,—the Saracens, possessing the virility of a new and victorious people, became the custodians of scientific thought with their natural taste for learning and absorbing new ideas whether it be in poetry, philosophy, or mathematics. They had further the

* Probably, the references in many of the Arabic texts of Geometricians and Arithmeticians apply respectively to the Greeks and the Hindus.

good fortune to possess rulers, who took pride in demanding intellectual, rather than commercial, treasure from the conquered people and were munificent patrons of learning whose courts were often adorned by scholars of different countries, irrespective of caste or creed. Not the least of the services of the Saracens to science consisted, as Cajori says, in that they adopted the learning of Greece and India and preserved what they received with care. When the love of science began to make itself felt in Europe, they readily transmitted the ancient learning to the Western countries. Possessing, as they did, an empire, which was 'an ellipse of learning' surrounding the Mediterranean basin with one focus in Europe and another in Asia, they became easily the connecting link between the East and the West and spread knowledge from cultured Asia to Mediæval Europe where learning was at its lowest ebb.

It was probably in the twelfth century, through Leonardo's monumental works that the Hindu science definitely passed to the Europeans from their Saracen masters. But even before the scholar, there are evidences to show that the busy travellers and merchants travelling all the great trade-routes from the East to the West and back must have carried with them a knowledge of number systems used in recording prices or in reckoning in the market and spreading them in different countries. Thus, no numeral system of the world could long have remained isolated and a convenient notation like the Indian one soon attracted the attention of the commercial world. even before any scholar attempted to write a treatise on it.

As early as the tenth century and even before, a form of Indian numerals without the zero, called Gobar or dust numerals was current in Europe, and Gerbert, a French scholar and monk of the tenth century, speaks of these numerals, though he could not appreciate them and employ them in the place of the Roman forms. The one improvement which was effected by the knowledge of Gobar numerals in Gerbert's time was the substitution, in the abacus, of apices marked 1, 2, 3, etc. in the place of as many Roman jecons or counters. We are told that the name 'apices' adhered to the Hindu-Arabic numerals (so called because they had their origin in India and were transmitted to Europe through the Arabs) until the sixteenth century and that the several names given to the figures indicate their Semitic origin.

Between the time of Gerbert and that of the appearance of the *Liber Abaci* of Leonardo, based on Musa's Arithmetic, there were two opposite schools of reckoning, one of them advocating the abacus and calling the other notation by the nickname of 'algorithma ciphra' (*i.e.*, a useless notation, because it involved the zero, a thing having no value). Indeed, the zero was unnecessary on the abacus, which, in my opinion, is no more positional than the notation current among the Burmese even to-day. For, where is the difference between writing 300105 as the Burmese do or $\frac{100}{3}\ \frac{10}{1}\ \frac{1}{5}$ as in the abacus for '315'?

The contempt which the abacists had for the zero is an indication that they, when left to themselves, would never have invented any symbol for zero. It must be impossible for a European abacist to conceive of such a tangible thing as one of their apices being set apart to denote 'nothing' or 'a blank' in the abacus.* This historical fact proves clearly the falsity of the premise 'The value of position and the invention of the zero are so obviously derived from the abacus', on which rests a good deal of Mr. Kaye's argument for the non-Hindu origin of the modern numerals.

In the eleventh century, a new activity in religion came about and with it a new interest in the algorisms, chiefly through the introduction of Arab learning. Arabic works were translated and contributions on arithmetic explaining the new algorism were made by a prominent Spanish Jew called John of Seville (?-1157 A.D.) and also by one Gherard of Cremona (1114-1187 A.D.). In the twelfth century the Englishman Adhelard of Bath translated into Latin Alkhowarizmi's Astronomical Tables from Arabic and Robert of Chester translated Alkhowarizmi's algebra. These men must thus have become familiar with the numerals that the Arabs were using. In the same century, one Rabbi Abraham wrote *Sefer ha-Mispar*, the Book of Number in the Hebrew language. In this book the Hebrew alphabet[39] with place-value is used for the numerals and a circle for zero; and the author acknowledges the Hindu origin.

But the greatest impetus that was given to the spread of the Hindu numerals in Europe was due to the great Italian mathematician Leonardo Fibonacci, who was born in the golden age of Pisa, when it was at the zenith of its commercial, religious and intellectual prosperity. Leonardo was a great traveller who had visited Egypt, Syria, Greece and other countries round about the Mediterranean, met scholars and merchants and imbibed from them a great deal of the numeral lore. He regarded all other numeral systems almost as errors (*quasi errorem*) compared with that of the Hindus. After his return to Pisa, he wrote his *Liber Abaci* in 1202 and rewrote it in 1228. The work was too difficult and learned for the merchants and too novel for the conservative Universities, while the times were unfavourable for the easy spread of knowledge. Still, as Pisa was a great intellectual centre drawing diverse foreign students to Italy—Bohemians, Poles, Frenchmen, Germans, Spaniards and others—the knowledge of Leonardo's text could not fail to spread.

*** It is possible, however, for a Hindu philosopher to do so; for the world of material objects is to him a huge nothing, an unreality, a hallucination due to 'ignorance' (अज्ञान). The converse process of denoting 'nothing' by a material symbol is quite in a line with his mental attitude.**

Meanwhile the popular treatises of Alexander de Villa Dei and John of Halifax did their share of the work in introducing the new numerals to the common people. It was probably due to the extended use of these popular treatises, especially that of John of Halifax (otherwise known as Sacrobosco), that the term Arabic numerals became common. In Sacrobosco's work, this science of reckoning is attributed to a philosopher Algus and reference is made to the Arabs as the inventors of this science. While some of the commentators, notably Petrus de Dacia, knew of the Hindu origin, they left the text as it stood and thus it came about that the Arabs were credited with the invention of the system.

Though the new numerals were fairly well-known in Europe by the thirteenth century, they had to wait till the sixteenth century to be generally used in business and in the schools. But they were used from time to time in dating manuscripts and in monuments. For various details in the development of the forms of the new numerals in the various parts of Europe during the last thousand years, we would refer the reader to G. F. Hill's work—'The Development of the Arabic Numerals in Europe, Oxford 1915'.

Only one point, however, deserves mention, *viz*., the use of the various anomalous forms * in Europe during the transition period from the Roman to the so-called Arabic numerals. For example, in the fifteenth century, we have

(*i*) a mixture of the Hindu and the Roman numerals:

E.g. MCCCC 811=1482.

(*ii*) a mixture of the positional and the non-positional notations, *e.g.* 12901=1291; in a manuscript of the Plimpton collection (*vide* D. E. Smith's *Rara Arithmetica*, p. 444) the date 1000 300 80 4 is given for 1384.

Such anomalous forms as the above have their parallel in India also. In Buhler's *Indian Palæography*, an instance is quoted of a Chicacole copper-plate inscription where the year 183 is represented by the Brahmi symbol for 100, the decimal 8 (for 80) and the syllable लो (for लोक=3) and the day of the month is given as '20' in decimal figures.

Such parallelisms as the above point the moral to such critics as Mr. Kaye † that human nature is the same everywhere and the differences that may be observed are not more than skin-deep.

(*To be continued.*) 115-133

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* For examples of the mixture of 'Arabic' with 'Roman' numerals, *vide* pp. 468, 469, *The Mathematical Gazette*, Vol. XII, No. 178, October 1925.

† Critics like Mr. Kaye make much of such parallelisms and rush to the conclusion that the later of two parallel developments is a copy of the earlier (*vide* p. 97, Cajori's *History of Mathematics*, 1919). Numerous instances of such prejudiced arguments advanced by Mr. Kaye are reviewed in the next chapter.

THE HINDU ARABIC NUMERALS.

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(Continued from Vol. XIX, No. 1.)

CHAPTER VI.

A Review of the Evidences regarding the Indian Origin of the Modern Notation with Place-Value and Zero.

As Mr. R. C. Dutt has put it, the history of Ancient India is a history of thirty centuries of human culture and progress and ancient Hindu literature takes us naturally far beyond the golden age of Greece. The earliest effusions of Hindu thoughts, ideals, and speculations are preserved in that monumental work, the Vedas which are considered to be the highest authority among the Hindus for all time; and it is remarkable to find what excellent precautions have been taken from time to time to prevent these records from corruption and interpolation, by means of a system of checks and counter-checks such as the following :—

(*i*) Special injunctions that knowledge should be learnt only from a Guru and not directly from the texts. This is probably due to the fear that texts may be corrupted or misread, while a Guru may be expected to transmit true knowledge.

(*ii*) More importance was given to *swara* and proper pronunciation than to meanings in recitations, the phonetic changes being recorded from time to time accurately in the Pratisākya-sutras.

(*iii*) The metrical form conforming to fixed laws which render alterations difficult.

All these give us a vivid picture of the scrupulous care with which the ancient texts have been preserved in India—a feature unparallelled in the histories of the other nations. The same scrupulousness, as has been pointed out already (*vide* p. 31 *supra*), prompted the early Hindus to invent the alphabetic and the word-numeral notations for numerals.

The development of the science of language, especially grammar, is also unique in India and dates back to some centuries before the Christian era. Witness the magnificent edifice of Sanskrit grammar due to Panini, the greatest grammarian that the world has produced. In Albrecht Weber's words, ' Panini's Grammar is distinguished above all similar works of other countries, partly by its thoroughly exhaustive investigation of the roots of the language and the formation of words; partly by its sharp precision of expression.........This is rendered possible by the employment of an algebraic terminology of arbitrary contrivance, the several parts of which stand to each other in the closest harmony, and which, by the very fact of its sufficing for all the phenomena which the language presents, bespeaks at once the marvellous ingenuity of its inventor and his profound penetration of the entire material of the language.' Here we have an indication of the remarkable aptitude of the Hindu mind for algebraic symbolism with its elegant conciseness, and this is well manifested in the *Dasagitika-sutra* of Aryabhata, which embodies the astronomical tables in a peculiar algebraic notation (*vide* p. 20 *supra*).

Further, in one of the Buddhist sacred books, the *Lalitavistara*, the hero Buddha is made to give out a scheme of names for large numbers, which go as far as 10^{53} and which the hero is prepared to extend unto Mahakalpas by a scale of orders of infinity (असंख्या) 'which is the tale of all the drops that in 10,000 years would fall on all the worlds by daily rain'. This reminds us strongly of the sand-reckoner of Archimedes (*vide* pp. 227-229, *The Work of Archimedes*, by T. L. Heath. C. U. P. 1897).

When the Greeks could devise a sand-reckoner with their traditional names of numbers not extending beyond a myriad (*i.e.*, 10,000), it is no wonder that the Hindus could think of scales of big numbers when they had regular traditional names upto 10^{18}. Again, surprises of genius are not uncommon in India. A Ramanujam in the twentieth century, without any proper training worth the name, was able to dream of problems which it had taken a hundred years for the finest mathematicians of Europe to solve and of which the solution is incomplete even to-day (*vide Proceedings of the London Mathematical Society*, Vol. 19, second series). When such has been the case, is it difficult to believe that the sand-reckoner of Archimedes could have been anticipated by a genius like Buddha, who was destined in later years to preach a religion which, of all religions, has the greatest number of adherents and which has influenced the morals and given spiritual comfort to hundreds of millions? (*Vide The Travels of Fahien*, translated by H. A. Giles, C. U. P., 1923.)

We may also mention that before the Christian era, there existed a tract on astronomy forming the sixth and the most important limb of the Vedas, which gives us an idea of the number work of those early ages (*vide* Monier Williams' *Indian Wisdom*, pp. 144, 177). In a very early document of the Hindus, the Sulva-Sutras in which practical methods are devised for the construction of altars to please the immortals, we find remarkable evidences of mathematical logic and acuteness displayed. It is specially noteworthy how our ancient Acharyas tackled the two kinds of irrationals $\sqrt{2}$ and $\sqrt{\pi}$

by rational approximations, impelled, as Thibaut admits, by the earnest desire to render their sacrifice in all its particulars acceptable to the gods and to deserve the boons which the gods confer in return upon faithful and conscientious worshippers.

All this implies a considerable advancement in arithmetic in very early times. Since the appearance of Mon Schrœder's important work *Indiens Literatur and Cultur* (Leipzig, 1887), the old view due mainly to Cantor that Indians owe all their mathematics to the Greeks is getting superseded by the sounder opinion that Hindu geometrical theory and conception of irrationals, etc., are entirely original, despite the unwarranted insinuations of Mr. Kaye in his article 'The Source of Hindu Mathematics', *J.R.A.S.*, 1910.

The above is a brief review of the original achievements* of the Hindus in several directions in very early times and although it does not bear directly on the origin of our numerals, yet it is highly relevant as showing the aptitude of the Hindu for mathematical and mental work of no inferior order.

The Hindu origin of the modern numerals with place-value is very likely; but we have no definite documentary evidence for it. We are entirely in the dark about their early inception; but we are more or less on safe ground as regards their development in India, the approximate period of the conception of the place-value, and the spread of the numerals through the Arabs to Europe.

Ages ago, suggestions for the forms of these numerical symbols may have been received from the Egyptian, Phœnician, or Chaldæan sources; but any attempt to develop a consistent theory regarding their foreign origin, Semitic or otherwise, would be only as futile as the several fanciful hypotheses discussed by F. Cajori on the origin of the numeral forms (*vide The Mathematics Teacher*, March 1925). Theories are not wanting, however, to derive the Indian numerals from—

(1) the order of letters in the ancient alphabet,

(2) the alphabetical expressions of certain syllables called अक्षर which possessed in Sanskrit some fixed numerical values,

(3) the first nine letters of the Greek alphabet (*vide The Mathematical Gazette*, July 1925),

and so on. But none of these conflicting theories give any satisfactory solution and, indeed, Messrs. Smith and Karpinski state that upon the evidence at hand, we might properly feel that everything points to the numerals as being substantially indigenous to India. We may notice also

* For a highly authoritative exposition of the achievements of the Hindus in the positive sciences, *vide The Positive Sciences of the Ancient Hindus*, by Dr. Brajendranath Seal.

some characteristic features in the development of the Indian notation which go to prove the same fact.

The early origin of the numerals must always remain obscure for two reasons :—

(1) The development of the notation has been very slow, extending over several centuries; and there are no authentic histories available earlier than the commencement of the Muhammadan era, *i.e.*, 622 A.D., while there are evidences to show that the place-value notation in some form was current earlier than this date.

(2) The positional idea may have flashed in a moment of truly divine inspiration to some unknown Hindu genius, who with his characteristic humility did not care to associate his name with it as its inventor but ascribed it to god.

Mr. G. R. Kaye is not tired of mocking at this Hindu trait (*vide* his article 'Old Indian Numerical Symbols' in the *Indian Antiquary*, Vol. XL, p. 49, and his book *Indian Mathematics*, p. 31) and makes us understand that the earlier investigators on the subject of numerals were misled by the orthodox Brahmin opinion that 'the invention of the nine figures with the device of places to make them suffice for all values being ascribed to the beneficent Creator of the Universe' (quoted from Krishna's commentary of Bhaskara's *Lilavati*—*vide* Colebrooke's *Translation of the Lilavati*).

In an article entitled 'New Light on Our Numerals', Mr. Ginsburg wrote, in 1917, "that our common numerals are of Hindu origin seems to be a well-established fact and that Europe received them from the Arabs seems equally certain but, how and when these numerals reached the Arabs is a question that has never been satisfactorily answered." A new light had been thrown on this question by Mr. M. F. Nau's publication in the *Journal Asiatique*, of an important fragment of Sebokht's writings in which there is a direct reference to the Hindu numerals. But Mr. Kaye has no faith in Nau's evidence (*vide* his *Indian Mathematics*, p. 31).

Severus Sebokht of Nisibus belonged to the second half of the seventh century and was a distinguished scholar in philosophy, mathematics and theology. He was the head of a convent in Nisibus, a great commercial centre and had numerous pupils through whom his knowledge could have been transmitted to other scholars all over Syria. Remembering that Syrian scholars were employed by the Caliphs as translators and educators we could easily understand how the Syrians could have imparted the knowledge of the Hindu numerals to the Arabs along with other facts relating to sciences. But how did Sebokht come to know of the Indian numerals? Since the exchange of goods and the exchange of ideas always went together, the trader and the traveller were busy agents in the transmission of numerals from the East to the West and it is not unreasonable to surmise that in an important centre like Nisibus of a very extensive trade, different systems of numeration were known and attracted the attention of an intellectually alert man like Sebokht. Of the Hindu numerals, Sebokht speaks in the following high terms of praise :—

" I will omit all discussions of the Science of the Hindus, a people not the same as the Syrians. . . . . their valuable methods of calculation and their computing that surpasses description. I wish only to say that this computation is done by means of nine signs. . . . ."

Among other evidences of the Indian origin of the decimal notation with its zero, we have already referred to—

(1) The use of 'Sunya' in the *Suryasiddhanta* and the works of Varahamihira.

(2) The variants of position arithmetic found in the word-numeral notation of Brahmagupta and the Katapayadi scheme of Aryabhata, the younger, such variants being most likely to occur only in the country of its (position-arithmetic's) origin.

(3) Subhandhu's mention (in the sixth century A.D.) of the zero-dot-symbols in a simile describing the stars.

(4) The reference to numbers taking different values according to their position, found in Vyasa Bhashya of Patanjali's *Yoga-sutra* which cannot have been composed later than 600 A.D.

But an argument against the Hindu knowledge of these symbols is that the Arabs about 700 A.D. did not know of them, but looked upon them as a strange invention when they were introduced to them in 776 A.D. Since the Arabs had just then come to rob India of her wealth and had yet no idea of plundering her culture, it is no wonder they did not know of the symbols. We can easily imagine a parallel instance in modern times of a lay traveller in civilized Europe not knowing anything about the modern theory of Relativity. It takes certainly some time for the latest discoveries and inventions to trickle down to the level of comprehension of the common herd of people.

As regards epigraphical instances (instances in copper-plate land grants) of the use of the nine-symbols with zero, there is some doubt. Dr. Fleet in the *Indian Antiquary*, Vol. XXX (p. 205), holds that many epigraphical forgeries (since the copper-plates were deeds of property) were made about the end of the eleventh century. This accounts for Mr. Kaye's sweeping remark that epigraphical evidence is the most unreliable so far as India is concerned (*vide J.R.A.S.*, 1910). But Colebrooke takes a more rational view and points out that the value of the evidence is not on that account totally invalid, since a successful forgery has to imitate the writing of the period in question adhering to the then current notions and traditions.

Some of the most important of the several epigraphical instances of the use of numerals given by such high authorities as Buhler, Kielhorn, Bhandarkar who are entitled to our greatest respect are quoted below :—

(1) Gurjara inscription of Chedi Samvat 346 (595 A.D.) contains the oldest epigraphical use of alphabet numerals with place-value. (Buhler.)

(2) A Pathari pillar inscription of Vikrama Samvat 917 corresponding to 861 A.D. A copper-plate (Kadab-plate) inscription, dated Vikrama Samvat 813 (756 A.D.). (Kielhorn.)

(3) A stone inscription of 815 A.D.; Dholpur stone inscription of 842 A.D. (containing the date in word-numerals) ; another inscription incised on a pilaster, dated 798 A.D.

(Dr. D. R. Bhandarkar.)

If we find the numerals in inscriptions as early as about 750 A.D., the system must have been in existence at least one or two centuries earlier. Even in Europe, it was only two hundred years after the introduction of the numerals that they began to appear on inscriptions and coins. Even Thibaut assures us that the Indian origin of the system now in use cannot be doubted. The united judgments of these scholars point to the rise in India of the modern system with place-value as early as 600 A.D.† The only dissenting voice is that of Mr. G. R. Kaye whose hypotheses we shall discuss presently.

Mr. G. R. Kaye's Refutation of the Indian Theory. Re-examination of his Hypotheses and Arguments.

The following statement in *J.A.S.B.*, Vol. III, No. 7 (1907), gives in a nutshell Mr. G. R. Kaye's position regarding the origin of the modern notation.

'The character of the Indian scripts, the evidence of inscriptions, the nature of the early notations in use among the Hindus, the nature of their mathematical works; the very custom at the present time among the Hindus who work on purely indigenous lines point to a foreign origin of the modern notation as probable ; while the foundations of the arguments of those who believe in an Indian origin are now shown to be either absolutely

† In Vol. XIX, p. 867, *Encyclopaedia Britannica*, Eleventh Edition, Prof. W. Robertson Smith writes 'What is quite certain is that our present decimal system, in its complete form, with the zero which enables us to do without the ruled columns of the abacus, is of Indian origin.'

unsound, almost unreliable; and consequently the Indian theory, if it is to stand, must be re-stated.'

Mr. Kaye claims to show that the premises utilized by such eminent orientalists as Chasles, Peacock, Woepecke, Cantor, Bayley, Buhler and MacDonell are all unsound and that their inference from such premises that the modern Arithmetic notation is of Indian origin is also untrue. He asserts that they were all misled by the boasting claims of antiquity (for their knowledge) put forward by the Brahman commentators. But he contradicts himself elsewhere (*J.A.S.B.*, p. 813, 1911) by noting that the Hindus never claimed to have invented the new notation and adds 'neither did they claim originality in Mathematics'. According to him, Bhaskara often speaks, with disdain, of Hindu mathematicians and refers to certain 'ancient teachers' as authorities. In Kaye's logic, however, these ancient teachers, not being named specifically, must have been the Greeks. Thus his arguments are very subtle and he shifts his ground quickly and imperceptibly. His misrepresentations (to quote his own expression) are 'all the more dangerous by appearing less startlingly false'. He is skilful in utilizing the statements of his authorities just to such an extent as will be favourable to his pre-conceived theories.

One of the fundamental principles of our critic is this. To accept anything as of Greek origin, any remote analogy is sufficient, while to show that anything is of Indian origin, numerous unmistakable evidences must be produced. Thus the slight resemblance of the Bakshali symbol [38] to the Greek symbol [39] used by Diophantus* is enough to betray the Greek source, while, however patent and obvious the reference to Hindu sources in Greek or Arab writings, it is not sufficient to warrant the inference of a borrowing from India; for, in such a circumstance, Mr. Kaye is prepared to misread and misinterpret the texts as in the case of the Arabic words Hindisa, Hindi, Hindasi and give other† meanings to these words, blaming encyclopædias and dictionaries for not giving that interpretation which will suit his purpose.

The most favourite and frequent of Mr. Kaye's fallacies is his generalization from one favourable instance. Since Dr. Fleet has changed the order of the figures in गिरिरसवसु into 'vasus, flavours and mountains', our critic generalizes, from this instance, that copyists have always had the tendency to adapt

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* As Dr. Heath suggests, the symbol of Diophantus is not really the inverted letter (39) but the uncial combination of ΛΙ into ⋔ and it is absolutely rash to connect ⋔ with + of the Bakshali manuscript. Further Diophantus places this sign before the quantities to be subtracted, whereas in the Bakshali manuscript, the sign comes after the number to be subtracted.

† *Vide* G. H. Ojha's comment on Mr. Kaye's interpretation of the terms Hindisa, etc., in *The Palæography of India*, p. 119.

notations to the system in vogue in their own times ; and hence, he would not trust anything but really first-hand evidence on which alone his conclusions should be based. But, is our critic really giving us a first-hand instance, when he quotes an example of a Greek notation of the time of Herodotus ?

We shall now take up, in order, the hypotheses on which Mr. Kaye relies for his non-Indian theory.

Mr. Kaye presumes that it would be natural to expect number-words and symbols to be affected by the mode or direction of the writing. For example, it would appear strange to see numerical symbols written horizontally in conjunction with a vertical script. Since the numerals (in the Indian notation) increased in value towards the left and not towards the right, 'the notation was introduced into India as it was into Europe from a right to left script'. On this hypothesis, Mr. Kaye points out the error in Burnell's translation of 'अङ्कानांवामतोगतिः' into 'the order of the letters (*viz.*, numerals) is from right to left'. To Mr. Kaye, the order of the letters is the order of the script (*i.e.*, from left to right in the Indian script); but the numbers are always expressed with the smaller elements first and not, as is the custom now, with the higher elements first.

Mr. Kaye's hypotheses are partially right while his inference is absolutely wrong. In numeration, the natural order is that of counting, *i.e.*, proceeding from the smaller to the bigger numbers. This is consistently followed in Indian numeration in which the smaller elements come first, *Ex.* एकविंशतिः = one and twenty. Relics of this form are found also elsewhere ; for instance, the French Quatre-vingte = four times twenty. But, as pointed out elsewhere the practice in notation has in almost all countries been to put the bigger elements (probably on account of their greater importance) before the smaller ones. Now, the terms 'before' and 'after' are relative to the script in use. Thus in a right to left script, the bigger element would be written naturally to the right of the smaller, while in a left to right script, the bigger element would be written to the left of the smaller. We have actual illustrations† of such a use, which confirm our view (*vide* pp. 2, 3 *supra*). Mr. Kaye himself quotes instances of the Hebrew notation (2) found on coins ; in which the higher order numbers occur first in the right to left order. He is unable to explain the mystery of this notation, for which we have here found a key. Again, when the Hebrews as well as the Arabs adopted the other order (*i.e.*, the Indian (40) order) which is inconsistent with their script, it must be clearly due to the influence of the Indian numerals. An explicit mention of such a rule as 'अङ्कानांवामतोगतिः' was found necessary to point out the difference between the orders of numeration and

† *Vide* pp. 112, 113. *The Palæography of India*, by G. H. Ojha (2nd Edn., 1918).

notation in India. In Europe, however, this difficulty was solved by changing the original order in their numeration and adapting it to the new notation. Thus we have evidences of two different kinds of numeration in the English language, one set of numeration from thirteen to nineteen and another kind from twenty-one onwards.

Mr. Kaye wishes to point out also that the bigger element coming to the right of the smaller would be the more convenient and natural order and refers us to Perry's *Practical Mathematics* and to works on the 'Theory of Numbers'. This order is, of course, (convenient and even) absolutely necessary when the highest order number is undetermined and indefinite as in : $a + br + cr^2 + \ldots\ldots$ But in Arithmetic, where the digits have definite values, and the higher orders more significant and important than the lower ones (as for example when we give the population of a country to the nearest million) the present order is naturally more convenient. The convenience becomes markedly obvious in such continued processes as division or extraction of square-root, where the movement from left to right will be appreciated by those accustomed to the left to right script.

We shall next examine Mr. Kaye's view of the epigraphical evidences.

The earliest epigraphical instance of the new notation quoted by Dr. Fleet is Saka 867 (A.D. 945-946). But according to Dr. Linders, the earliest is the stone inscription of Dholpur, dated Vikrama Samvat 898. While Buhler's Chicacole inscription of 641 A.D. is now known to be spurious, Dr. Linders and Dr. Fleet doubt the Kadab inscription of 813 A.D. Mr. Kaye himself has come across only two instances of the symbolic words of the ninth century, three of the tenth and a few of the eleventh, but numbers of later date. This gives rise to suspicion in Mr. Kaye's mind and Mr. Damant is quoted as saying : 'The practice does not seem to be one of very great antiquity and many of the supposed older dates are doubtful' (*vide Indian Antiquary*, VI, 13). In this way, seventeen of the earliest Indian epigraphical instances due to Fleet, Kielhorn and others are quoted and all of them rejected as worthless on a policy similar to that adopted by the wise Caliph Omar who gave orders to the burning of the Alexandrian Library.

The epigraphical records are believed by Mr. Kaye to be either ingenious forgeries or wrongly interpreted by epigraphists who assumed that the new notation was common in India much earlier than the ninth century. Mr. Kaye would require the epigraphists to re-interpret these records with a contrary assumption in their minds.* Meanwhile, he would hypnotise himself into the strong conviction that the figures given either stand before him in a suspiciously modern form or do not allow of any direct interpretation, or are proved to be spurious, or that the record shows some signs of being tampered with. Even the unsuspicious dates such as the Bagumara Inscription of A.D. 867 (*vide Indian Antiquary*, XII, 181, XVIII, 56, and XXIII, 131), by the very reason of their uniqueness, call for explanation. After examining all these evidences, Mr. Kaye could conscientiously come to the only conclusion that to the eleventh century only we should turn for evidence of the use of the modern system of notation in India.

We shall next turn to the historical evidences. Mr. Kaye, as usual, laments that the so-called historical evidence is of so little avail for him. In his opinion, even such a reliable investigator as Alberuni says little pertinent to the question in hand, even though he mentions in unmistakeable terms (*vide* Alberuni's *India*, Vol. I, pp. 174 and 177).

'The Hindus do not use the letters of the alphabet for numerical notation as we use the Arabic letters in the order of the Hebrew alphabet. The numeral signs which we use are derived from the finest forms of the Hindu signs.............. I have studied the names of the orders of the numbers in various languages...........................and have found that no nation goes beyond the thousand. The Arabs too stop with the thousand....................

The Hindus use the numeral signs in Arithmetic in the same way as we do. I have composed a treatise showing how far, possibly, the Hindus are ahead of us in this subject.'

In the face of such plain confessions of Hindu superiority at least in arithmetic, from one who has frequently spoken of the Hindus in terms of contempt, Mr. Kaye tries to utilize to his sinister logic an exaggeration of Alberuni that the Hindus* he came across did not know the fundamental principles of mathematics and says that Alberuni's statement must be read in the light of this. Further Mr. Kaye doubts whether Alberuni was in a better position to judge of the Hindu numerals than Canon Taylor who is certainly wrong in his conclusion. We believe, however, that Alberuni was certainly in a better position, for he came in living contact with the Hindu numerals actually in use about eight hundred years before Canon Taylor and must have, therefore, been in possession of much better and more reliable evidence than Canon Taylor could get in his time.

It is interesting to see how Mr. Kaye disposes of the tradition of an Indian origin that existed among the Arabs. Now he adopts his usual trick of viewing an authority as sound or unsound according as he does or does not

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* I doubt whether Alberuni really came across Hindu mathematicians of repute. He gives a list of word numerals in use in his time and among the symbols for ten, he includes ignorantly खेन्दु as if it were one word. (खेन्दु=ख+इन्दु.)

This is one indication that he could not have picked up his knowledge invariably from authoritative sources.

fall in with his pre-conceived notions. He accepts Wopecke's authority when he (Wopecke) says we cannot attach any value to such references found in the commentary on the Talkhis of Ibn Albanna and a commentary by Husain Bin M. A'lmahalli on a work by Abdul Kadir Alsakhwi as follows:—

" A Hindu took some fine powder, spread it on a table and made upon it certain calculations and then put it away for future reference. " But he rejects the same authority as unsound when he translates ' Hindasiyyat ' as Indian, Mr. Kaye would like him to interpret it as 'geometrical' whether it gives sense or not in the context. He contends, in this connection, that the regular *duorum falsorum* or operation of the balance is derived geometrically and therefore Ibn el-Benna's explanation should be interpreted, ' As to the balance, this procedure is a geometrical (not Indian, though the Arabic word admits of both interpretations) method.' In support of his interpretation, a geometrical elucidation of this principle due to El-Sabi is quoted. El-Sabi proves in the manner of old Euclid that if the line ab is divided into three parts ag, gd, db, then $ab.gd + ag.bd = ad.bg$ and tries to deduce from this the principle in question. I fail to see any connection between the two results except perhaps some remote analogy. The explanation is really unsound; properly speaking, the principle of proportion must have been used to explain the rule. But the early Arabs were not quite proficient in proportion as evidenced by Musa's method of finding the length of a side of a square inscribed in an isosceles triangle (*vide* his Algebra translated by F. Rosen); while it is interesting to add that though the Hindus have not mentioned (so far as the extant texts go) the rule in the particular form in which it is found among the Arabs, the substance of the rule has been more scientifically utilized by Aryabhata in his mensuration of the trapezium and by Brahmagupta and Bhaskara in the rules of interpolation connected with their sine-tables.

Again, in another connection, where Ibn Sina relates the properties of squares and cubes (*viz.*, the square remainders are respectively 1, 4, 9, 7 while the cube remainders are 1 and 8), Mr. Kaye asserts that the interpretation 'geometrical' is sounder on the strength of a geometrical proof* devised by himself. It is an elementary mathematical platitude that with a little ingenuity many elementary theorems (in mathematics) can be graphically or geometrically demonstrated and on that account alone any arithmetical or algebraic theorem should not be called geometrical.*

* One of the finest theorems in Higher Arithmetic is the Legender's Law of Reciprocity which is proved partly geometrically. There are several such instances of geometrical investigations in the Theory of Numbers. But the theorems on that account are not called geometrical. (*Vide* Mathews' *Theory of Numbers*, pp. 41, 42 and Chapter IV, Binary Quadratic Forms: Geometrical Theory.)

Apart from the reference to Indian numerals in Arabic texts, certain mediæval works also contain unequivocal mention of the modern system of arithmetical notation as Indian. Mr. Kaye himself admits the difficulty in interpreting 'Indian' as 'geometrical' when the term is applied to numerical symbols. How then does he solve it? The Greek geometrical terminology for numbers comes to his rescue. The Greeks termed odd numbers gnomons, compound numbers oblongs, the product of two numbers plane, and the product of three numbers solid, while there were other numbers known as triangular, square, cube, polygonal, etc. There is, besides, the famous geometrical number of Plato. But nowhere do we find the appellation 'geometrical' in Greek texts applied as a generic term for any number. Only particular types of numbers have been called geometrical and it does seem certainly far-fetched to call all numerals geometrical. If the reasons for the wrong interpretations of the Arabic word into 'Indian' be due, as Mr. Kaye believes, to the following false premises:—

(*i*) the word cannot by any possible means imply 'geometrical' in the passage referred to,

(*ii*) a statement by Taylor in his Introduction to 'Lilavati',

(*iii*) the Arabs owe their knowledge of geometry* to the Hindus,

the reasons for the other interpretation 'geometrical' due to himself are based on another set of unreliable and unverified hypotheses:—

(*a*) the theorem referred to can be proved geometrically (as indeed any other elementary arithmetical theorem);

(*b*) the abacus was never in use in India;

(*c*) the Hindus owe their knowledge of arithmetic to a foreign source.

Thus the balance of wrong hypotheses is equal on both sides and we stand unconvinced by Mr. Kaye's eloquent outbursts of plausible reasoning.

Let us now record Mr. Kaye's view of the use of the abacus in India.

According to Mr. Kaye, the examples of the existence of abacus quoted by such writers as Warren, Bayley, Burnell are all too modern to be of any value. Burnell in his *South Indian Palæography* tells us that the Indian abacus was by using heaps of cowries for the numbers, the number of these shells being equal to that of the number expressed, the cipher (40) being a blank space. He adds "Warren in his *Kalasankalita* mentions a counter to express the cipher, but I have never found this to be done." Mr. Dikshit mentions the use of a wooden board called पाठी which is covered with dust

*I believe Mr. Kaye meant 'arithmetic' by the word 'geometry' and in the heat of the argument, made no distinction between the two words.

when used for purposes of reckoning; the numerals used while working on this plank were called by the Arabs Gobar or dust numerals. Alberuni's reference to the Indian custom of writing on the sand and the use of non-alphabetic numeral signs also indicates the use of some form of abacus. There is a relic of this ancient sand-writing custom even in modern times in the Burmese practice of writing on the ground in the dust or on black paraback. All these evidences are worthless and irrelevant to Mr. Kaye and he does not want to accept that one form of abacus was 'a tray containing sand which could be readily grooved with the fingers'. He imagines that there is a confusion between the terms 'abacus' derived from 'Abaq' (sand), 'Gobar' (powder) and 'writing in the dust'.

Taylor, Woepecke, Bayley, Burnell and others derive the Sanskrit word शून्य from the vacant space in the abacus. According to Taylor the word शून्य was translated into Arabic by the word 'Syfr' having a like meaning. Dr. Murray's *New English Dictionary* also confirms the derivation of the word 'Cypher' from the Arabic 'Syfr' and Sanskrit 'Sunya'. According to Mr. Kaye, all these authorities are unreliable, for it is very doubtful—

(*i*) whether the so-called Arabic numerals are really Indian;

(*ii*) whether the Arabs really received their numerical notation from India;

(*iii*) whether any form of abacus was in use at all in India.

In his logic, since there is no direct evidence* that the abacus was in existence in Ancient India, the Indians could not have invented the zero symbol also. In the present writer's opinion, the abacus is not a necessary and indispensable precursor of the zero. The early word numeration of the Hindus which gave, in order, the number of units, tens, hundreds, etc., in a number leads, naturally, to it. Further, history tells us how the abacists and the algorists were at logger-heads in Europe between the twelfth and the fourteenth centuries.† Especially, it is a noteworthy fact that the abacists had a contempt for the zero which is a clear evidence to show that left to themselves they would never have invented the zero symbol.

We shall next take up the evidences of place-value notation found in Indian arithmetical treatises and examine how Mr. Kaye handles them.

While he puts up such a fight for the interpretation of the Arabic word 'hindisa', he coolly ignores several explicit instances in Indian arithmetical treatises of using the word स्थानं (place) in connection with numerals. Rodet's translation of the term स्थानान्तरे in Aryabhata's text explaining the ordinary method of root-extraction :—

* Dr. Fleet's reference to गाणित्र (an instrument to reckon with) in *J.R.A.S.*, 1911, has been noted elsewhere.

† *Vide* Smith and Karpinski's *Hindu-Arabic Numerals*, p. 120.

भागं हरेदवर्गान्नित्यं द्विगुणेन वर्गमूलेन ।
वर्गाद्वर्गे शुद्धे लब्धं स्थानान्तरे मूलम् ॥

by 'à distance d'une place' or 'à intervalle d'une place on d'un rang' is wrong in Mr. Kaye's opinion. Mr. Kaye gives us a warning that we should not be misled by the commentators who came very much later than the original writers (and knew the decimal notation), while he himself misleads us by quoting a problem from Iamblichus (360 A.D.) as affording a distinct proof that the Greeks had perfectly clear ideas of the value of position (unlike the Hindus).

The problem as stated by Kaye is this :—
' If the digits of any three be added together and the digits of their sum be added together and so on, the final sum will be six', which is, of course, mathematically wrong.* Mr. Kaye, I believe, purposely mistranslates the Greek text in order to give undue credit to the Greeks. The use of the word 'digit' is unwarranted. The word 'digit' has no meaning and no counterpart in the Greek non-positional notation. The correct text as translated more faithfully by Dr. T. L. Heath runs thus :—

' Take the sum of the three consecutive numbers the greatest of which is divisible by 3 ; this will consist of a certain number of units, a certain number of tens, a certain number of hundreds and so on. Take the units in the said sum as they are, then as many units as there are tens, as many units as there are hundreds and so on and add all the units so obtained together. Apply the same procedure to the result and so on. Then the final result will be the number 6.' The problem stated as above contains really no suggestion of the place-value.

The practice of adding certain units comprised in a number was common among Kabbalists who were attaching mystic significance to numbers ; thus ALSHDI, Alshaddai, or God Almighty is equivalent, when interpreted as Hebrew numerals, to 1, 30, 300, 4, 10 (or 345) and the units 1, 3, 3, 4, 1 being added up yield 12 and 1, 2 added together yield 3 suggesting the trinity in God. This shows that no digits were added but merely the number of units, tens, hundreds, etc., denoted by the alphabetic numerals forming a number. It is a far cry from the letters such as A, L, Sh, D, I comprised in a number to the notion of digits with a place-value

*Mr. Kaye fails to mention that the numbers must be consecutive (and not any three) of which the highest must be a multiple of 3. This error is probably unconscientious. But in other places, he indulges in wilful mistakes, which is quite unpardonable ; for instance (*i*) he says that the Katapayadi notation and Aryabhata's notation are alike, (*ii*) in interpreting राघवाय numerically, he wishes to suggest a kind of non-positional notation by putting र = 2, घ = 0, ...etc., (*iii*) the word numeral notation was introduced into India about the ninth century (*vide The Indian Mathematics*, p. 31).

and it will be certainly a foolish fancy to infer place-value notation from such instances.

Though we find a method of extracting square-root given in Greek texts similar to that of Aryabhata, yet, in no extant Greek writer do we find any description of the operation of extracting the cube-root* while it is a noteworthy fact that methods for finding cube-root are mentioned in all the Indian arithmetical treatises from Aryabhata onwards. This is a sufficient indication that the Indian methods developed independently of the Greek. The Hindus, with the advantage of their decimal notation and their aptitude for algebra could easily extend their method of finding the square-root to cube-root also; but the Greeks with their geometrical bent of mind and their non-positional notation, could not evidently proceed beyond the square-root for which they got the suggestion from Euclid.

It is a futile argument that the place-value notation could not have been in vogue in India, because even to the present day the Hindus taught on indigenous lines, learn tables of squares to a prodigious extent. Even to-day, we are using tables of squares, cubes, logarithmic tables and ready reckoners to facilitate our computation and the existence of such tables does not signify really a non-positional notation in use.

Another argument put forth against the Indian theory is that its use is not indicated in the rules for the fundamental operations given by Brahmagupta. But his use of the word 'गोसूत्रिका' (string for cattle) in connection with multiplication suggests that the digits of a number are written (possibly in separate compartments) in a horizontal row while the multiplier is taken into each of these digits and the individual products summed up finally. Brahmagupta suggests also some short method of multiplication (something like the familiar rule, to multiply a number by 99, multiply it by 100 and subtract the number itself) in the following verse :—

गुण्यो राशिर्गुणकार राशिनेष्टाधिकोनके न गुणः ।
गुण्येष्टवधोनयुतो गुणकेऽभ्यधिकोनके कार्यः ॥

i.e., if the multiplier be too great or too small (as compared with the intended multiplier) the multiplicand is to be multiplied by the corresponding excess or deficit and this product is subtracted from or added to the (original) product.

As the commentator remarks, this rule is intended to correct errors in cases where, by mistake, the multiplicand has been multiplied by a number too great or too small.

There is hardly anything in the above rules specially convenient for non-place-value notation, nor can one infer from them, circumstantially, that the place-value was not known to Brahmagupta.

**Vide Greek Mathematics*, by T. L. Heath, Vol. I, p. 63 and Vol. II, p. 341.

Another evidence adduced in support of the non-Indian theory is that among the few still extant old arithmetical practices, the old ideas of notation (non-positional) prevail as in the case of the Burmese Arithmetical operations pointed out by Sir R. Temple (*vide Indian Antiquary*, Vol. 1891). This is, in Mr. Kaye's mind, the proof absolute that the new notation is not of Indian origin. To this we shall reply "Why does not Mr. Kaye accept, reasoning on the same lines, that since in some parts of India, the abacus is still used, the abacus must also have existed previously in India ?"

Because in some remote corners of India, untouched by the spark of later inventions indigenous or otherwise, some old antiquated system is still in use, it does not follow that no improvements were effected later by Indians. India is a vast continent and it is no wonder that an invention or discovery made in one place has not yet penetrated some of the remote corners. Witness, even at the present moment, how a large part of India in remote villages is still a stranger to the civilization in towns. In India, we can see not one homogeneous civilization but a series of different levels of civilization belonging to different periods in the unfortunate checkered history of India, and to say arbitrarily that some one of these alone is a representative of ancient Hindu civilization or intellectual attainment is, of course, unfair.

Another reason trotted out against the Indian theory is Mr. Kaye's conviction that there was never a school of Indian Mathematics. This reminds one of how a distinguished British Mathematician described in 1816 in the *Encyclopœdia Britannica* (Art. Arithmetic) the *Lilavàti* as 'a short and meagre performance headed with silly preamble and colloquy of the Gods'. Colebrooke laments the negligence of the author, his want of research and reliance upon obsolete authorities and antiquated disquisitions. But Mr. Kaye cannot be accused of any of these defects but rank prejudice which has unfortunately blinded him to the true perspective of facts. He was led to this valuable opinion about Indian Mathematics apparently by the fact pointed out by Chasles* that Bhaskara and the commentators of Brahmagupta were not competent enough to appreciate an important theorem of Brahmagupta. This fact again has led him to another logical conclusion that Brahmagupta himself was of the same type as his successors.

One of the important evidences for the Indian origin of the decimal notation is the Bakshali manuscript which contains one of the most explicit references to the Indian numerals. Dr. Hoernle's arguments in favour of the antiquity of this manuscript are freely criticized by our critic as a vicious circle, and we beg to point out how our learned critic himself falls into

* Though Chasles' authority is relied on in this instance, his evidence is considered untrustworthy when he attributes to Brahmagupta the formula for the sides of a rational right-angled triangle.

another vicious circle which in effect involves the following untrue and improbable assumptions :--

(1) That Indian arithmetic and algebra are entirely of foreign origin.

(2) That the Hindus got their elements of arithmetic undoubtedly from Diophantus and also to some extent from the Chinese and the Arabs.

(3) That the principle of position-value was unknown in India till about the twelfth century and hence the Bakshali arithmetic should belong to a period later than the twelfth century.

Having thus disposed of many of the arguments in favour of the Indian theory as being baseless, Mr. Kaye doubts whether the Arabs really owe anything to the Hindus. In a recent book 'Arabic Thought and its Place in History' by De Lacy O'Leary, D.D. (Trubner's Oriental Series, New York, 1922), we read: "About 156 A.H., an Indian traveller brought to Bagdad a treatise on arithmetic and another on astronomy: the astronomical treatise was the Siddhanta which came to be known to the Arabic writers as the Sindhind, ... it opened up a new interest in astronomical studies ! . . . .
The Indian work on arithmetic was even more important as by its means the Indian numerals were introduced, to be passed on in due course, as 'Arabic' numerals and this decimal system of numbering has made possible an extension of arithmetical processes and indeed of mathematics generally which would have been difficult with any of the older and more cumbersome systems." Mr. Kaye doubts the veracity of such statements as the above, which, according to him, are based only on the authority of Colebrooke. Indeed he proclaims that Mohammed Ben Musa's mathematical work (which inspired Leonardo's treatises of the thirteenth century) was not based on Hindu originals and ridicules the poor translator of Musa's Algebra for quoting parallels from *Lilavati* (a later work than Musa's) for the ratio of the circumference of a circle to its diameter. He hints mischievously that F. Rosen might be one of those who believed that the circle could be exactly squared, because he had not given any credit to the Arabic comment (in the margin of Musa's Algebra) which ran 'they (the ratios corresponding to π) are approximations and not exact truths; for God alone knows what the exact truth is.' Mr. Kaye calls this a very brilliant exposition of the case and wants to give credit to the Arabic annotator for noting the irrationality of π (which, as we know, is a very recent discovery). It is probable that the above Arabic exposition was suggested by the Hindu value $\sqrt{10}$ (for π) which, of course, cannot be exactly evaluated.

Whether Rosen argued correctly or not, there is no doubt about the Hindu origin of Mu'sa's text ; for, how else could Musa have got the values $\sqrt{10}$ and $\frac{62832}{20000}$ (of π) just in the form in which they occur in Indian Mathematical works (*vide* Prof. Mitra's article 'The Ancient Hindu Knowledge of Mathematics', *Modern Review* (Vol. XIX, 1916) pp. 638, 639). Because Rosen, not being aware of Aryabhata's text, could not quote Aryabhata's value in support of his Indian theory and took the value from *Lilavati*, in Mr. Kaye's opinion, the entire theory falls to the ground. Mr. Kaye says 'It is not necessary here to take into account the value given by Aryabhata, as Brahmagupta on whose work that of M. Ibn Musa is said to be based, did not give it.' The point at issue lies between M. Ibn Musa, Brahmagupta and Bhaskara. For Mr. Kaye's purposes, if an upholder of the Indian theory makes a mistake or slip in one place, the whole theory becomes vitiated thereby, and unreliable.*

Thus, he mercilessly attacks the arguments of the early orientalists who were not in full possession of facts.

If Mr. Kaye could scent the contents of the lost works of Diophantus (*vide* p. 15, *Indian Mathematics*) in Brahmagupta's text, why should he not also with equal reason trace the contents (not traceable directly to Brahmagupta) of M. Ibn Musa's work to some lost Indian work prior to Brahmagupta? We know, for certain, on Brahmagupta's authority that the old text of Brahmasiddhanta had become very rare with many parts missing owing to lapse of time :

ब्रह्मोक्तं ग्रहगणितं महता कालेन यत् श्लथीभूतं ।

This constant reference to previous lost or nearly lost works in nearly every one of the extant early Indian works is entirely ignored by Mr. Kaye who makes such a fuss of the lost works of Diophantus.

Enough has been said in the previous pages to expose the bias and one-sidedness of Mr. Kaye's arguments and the methods adopted by him to support his creed. The way in which he rejects evidences and browbeats authorities favouring the opposite faith by exaggerating their weaknesses and foibles, his concoction of circumstantial evidences on imaginary grounds to strengthen his theory that everything mathematical is of Greek origin—these are all quite characteristic of the author. To do him justice, however, it must be said that he is one of those who cannot help believing that all culture, science and civilization had their origin in Greece. In the *Journal Asiatique*

* Another instance of this trick of dealing with authorities, which occurs in his *Indian Mathematics*, p. 31, is quoted below :—

According to M. Nau, the Indian figures were known in Syria in A.D. 622 ; but his authority makes such erroneous statements about 'Indian' astronomy that we have no faith in what he says about other 'Indian' matters.

M. F. Nau mentions how a distinguished scholar of Nisibus, belonging to the seventh century, hurt by the arrogance of certain Greek scholars who looked down on the Syrians, made the remark which is as true to-day as it was more than a thousand years ago.

'Science is universal and is accessible to any nation or to any individual who takes the pains to search for it. It is not, therefore, a monopoly of the Greeks, but is international. . . . . . .

If those who believe, because they speak Greek, that they have reached the limits of science should know these things (that is, the Indian invention of the numerals)* they would be convinced that there are also others (*viz.*, the Hindus)* who know something.'

In his *History of Mathematics*, F. Cajori mentions two other writers, Carrade Vau and Nikol Bubnov advancing arguments tending to disprove the Hindu origin of our numerals. But none of these arguments is sufficiently weighty to decide the question one way or the other.

Conclusion.

As the late Sir Asutosh Mukherjee has pointed out, our knowledge of early Indian mathematics is somewhat limited and fragmentary: 'There is no exhaustive collection of Sanskrit manuscripts on astronomy and arithmetic and the works which have been published or rendered into English form a very small proportion of what is known to have existed at one time. Under such circumstances, it is somewhat difficult to make a definite pronouncement on the subject of the indebtedness of Indian mathematics (in particular, of Indian notation)* to foreign sources.' Until new evidence of great weight can be submitted in support of the non-Indian theory, we have to believe from the evidence now available in the Hindu origin of the numerals or keep our minds open and await further important evidence in support of either theory.

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* The statement within brackets is explanatory and inserted by the present writer to make the context intelligible.

APPENDIX.

(1). $\begin{matrix} <\,< & \vee\vee\vee \\ <\,< & \vee\vee\vee \\ <\,< & \vee\vee\vee \\ <\,< & \end{matrix} = 89.$

(2). ʃYϭP = 1 + 80 + 100 ; ℸ1 = 8 + 10.

(3). LX = 50 + 10. (4). HΔ = 100 + 10.

(5). ωக = 10 + 1. (6). ϟΦζ = 100 + 80 + 9. (7). 733 = 10 + 20 + 20.

(8). 2ω = 2 × 10. (9). λ| = 100 × 1. (10). << V < = 10 × 10 × 1000.

(11). Γ = 5 × 100. (12). Ŧo = 1000 × 10. (13). ௱ரு௰ஆ = (100 + 5 × 10) × 1000.

(14). VV<V> <<<VV = 2 × 10 × 100 + 80 + 2. (15). ψNΔ = 700 + 50 + 4.

(16). ϡ Ͻ ℨ = 300 + 50 + 4. (17). ά ŋ φ ϙ β = 18592.

(18). ⅢⅠ ≟ T – Ⅱ = 99617. (19). X = 4. (20). ℸ = 10. (21). 3 = 20.

(22). λ| = 100. (23). XX = 8. (24). | (1) ‖ (2) + (4) Є or Ɛ (6) C or Ͻ (50) ⅄ (200) Ɲ (200) Ŧ (200)

(25). — (1) Ψ (4) φ (6) ∝ (10) ∘ (20) (26). 7 (7) 8 (9)

(27). φ = 80. (28). ϟ = 100, T = 1000. (29). Ŧ+ = 60,000; TϟΊ = 1700.

(30). ~ (1) ≈ (2) ≋ (3) 7 or Ⴗ (4) Ⴀ or ꟷ (5) φ (6) 7 (7) ∽ or ∽ (8) ρ (9) ∝ or ∝ (10) Θ (20)

ꭗ (40) Ꭹ (70) ∩ (100) ∩ (200) ɱ (500) 9 (1000) 9 (2000) 9 (3000) 9 (4000) 9 (5000) 9 (70,000)

(31). (F), C, S; Z, ζ (7); (B), H, η (8); θ, σ, ϖ (9). (6)

(32). Δ. (33). Ұ. (34). ϑ. (35). ꝗ.

(36). ∼ or ⌒ (1); 2 or 3 (2); 3 or ʒ (3); ५ (4); ч (5); ɿ (6); 2 (7);

ħ (8); ◌ (9); • (0) (37). ∝ = 10. (38). +.

(39) ϯ (40). PDϑ = 100 + 60 + 9. (41). ∴|·|∴ = 303.